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LETTER TO THE EDITOR

A modified regularized long-wave equation with an exact two-soliton solution

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Abstract. Numerical studies of the regularized long-wave (RLW) equation of Peregrine and Benjamin and colleagues

$$u_t + u_x + (6u^2 - u_{xt})_x = 0$$

suggests it has a two-soliton solution although an analytic form for this has not yet been found. We show that a modified form of the RLW equation

$$u_t + u_x + (4u^2 + 2w_x v_t - u_{xt})_x = 0,$$

with $u = w_t = v_x$, has an exact two-soliton solution. The modified equation has the same solitary-wave solution as the original equation and its analytic two-soliton solution agrees closely with the numerical solution of the RLW equation.

As a model equation describing a variety of interesting non-linear wave phenomena, the Korteweg–de Vries (KDV) equation

$$u_t + u_x + (6u^2 + u_{xx})_x = 0 \tag{1}$$

has received considerable attention in the literature (Scott *et al* 1973). An alternative model for non-linear waves, first suggested by Peregrine (1966) and Benjamin *et al* (1972) is the so called regularized long-wave (RLW) equation

$$u_t + u_x + (6u^2 - u_{xt})_x = 0. \tag{2}$$

The main advantage of the RLW equation as a model for non-linear unidirectional long waves is that an analysis of the linearized equation shows that the velocity of sinusoidal solutions $f(\kappa)\exp(\mathrm{i}\kappa x - \mathrm{i}\omega t)$ has the form

$$v(\kappa) = (1+\kappa^2)^{-1} \tag{3}$$

whereas for the KDV equation we find

$$v(\kappa) = 1 - \kappa^2. \tag{4}$$

Hence for large κ (short wavelength) the velocity of RLW solutions tends to zero whereas the velocity of KDV solutions becomes negative and unbounded.

A great deal is known about soliton and multi-soliton solutions of the KDV equation but no corresponding results are known about multi-soliton solutions of the RLW equation. The corresponding single solitary-wave solutions are

RLW: $$u(x, t) = \tfrac{1}{4}[a_1^2/(1-a_1^2)]\,\mathrm{sech}^2\,\tfrac{1}{2}\{a_1 x - [a_1 t/(1-a_1^2)] + \delta_1\} \tag{5}$$

KDV: $$u(x, t) = \tfrac{1}{4}a_1^2\,\mathrm{sech}^2\,\tfrac{1}{2}[a_1 x - a_1(1+a_1^2)t + \delta_1]. \tag{6}$$

However recent numerical work by Eilbeck and McGuire (1976) has shown that the RLW equation appears to have at least two- and three-soliton solutions, although their analytic forms are as yet unknown. The purpose of this letter is to show that although we cannot yet solve this problem, it is possible to construct an equation which is close to the RLW equation in some physical sense and yet has an exact two-soliton solution.

This new equation is required to have the following properties: (i) the same short wavelength properties as the RLW equation; (ii) the same solitary-wave (single-soliton) solution as the RLW equation; (iii) an exact two-soliton solution.

Such an equation can be constructed using Hirota's method of the dependent variable transformation (Hirota and Satsuma 1976a). Defining

$$\mathrm{D} = \frac{\partial}{\partial x} - \frac{\partial}{\partial x'}; \qquad \mathrm{T} = \frac{\partial}{\partial t} - \frac{\partial}{\partial t'}; \tag{7}$$

then the transformation $u = (\ln f)_{xx}$ transforms the KDV equation into

$$[(\mathrm{D}^4 + \mathrm{D}^2 + \mathrm{DT}) f(x, t) f(x', t')]_{\substack{x=x'\\ t=t'}} = 0. \tag{8}$$

A one-parameter solution of (8) is

$$f = 1 + \mathrm{e}^{\theta_1}; \qquad \theta_1 = a_1 x - \omega_1 t + \delta_1. \tag{9}$$

It is obvious that the form of the operator in (8) determines the relation between ω_1 and a_1, i.e.

$$a_1^4 + a_1^2 - a_1\omega_1 = 0 \Rightarrow \omega_1 = a_1(1 + a_1^2). \tag{10}$$

Putting $u = -(\ln f)_{xt}$ in the RLW equation we find the one-soliton solution of the RLW equation (5) can be put in the same form as (9) using the relation

$$\omega_1 = a_1(1 - a_1^2)^{-1}. \tag{11}$$

Since this can be rewritten as $a_1^2\omega_1^2 + a_1\omega_1 - \omega_1^2 = 0$ this suggests the appropriate modification of equation (8) is

$$[(\mathrm{D}^2\mathrm{T}^2 - \mathrm{DT} - \mathrm{T}^2) f(x, t) f(x', t')]_{\substack{x=x'\\ t=t'}} = 0. \tag{12}$$

Since for the RLW equation $f = \exp q$, where $q_{xt} = -u$, we get on substituting for f in (12)

$$u_t + u_x + (4u^2 + 2w_x v_t)_x - u_{xxt} = 0, \tag{13}$$

where $u = w_t = v_x$.

This is our modified version of the RLW equation, differing only in the term $2w_x v_t$ in the non-linear part. Since in all physical applications the KDV and RLW equations are obtained using perturbation theory (Benjamin *et al* 1972) from which the zeroth order term is $u_t + u_x = 0$, the modified version of the RLW equation is easily shown to be equivalent to both the RLW and KDV equations to the same order of perturbation ($u_t = -u_x \Rightarrow u^2 = w_x v_t$). The modified RLW equation (13) has the same solitary-wave solution as the RLW equation (2) and has the following two-soliton solution:

$$f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + A\,\mathrm{e}^{\theta_1 + \theta_2} \tag{14}$$

where

$$A=-\left(\frac{(a_1-a_2)^2(\omega_1-\omega_2)^2+(a_1-a_2)(\omega_1-\omega_2)-(\omega_1-\omega_2)^2}{(a_1+a_2)^2(\omega_1+\omega_2)^2+(a_1+a_2)(\omega_1+\omega_2)-(\omega_1+\omega_2)^2}\right) \tag{15}$$

and

$$\theta_i=a_ix-\omega_it+\delta_i; \qquad \omega_i=a_i/(1-a_i^2). \tag{16}$$

We have $A>0$ for all real a_1, a_2 and hence we can calculate the two-soliton phase shift Δ which occurs when the larger soliton passes through the smaller (figure 1)

$$\Delta=-\tfrac{1}{2}\ln A. \tag{17}$$

It is interesting that Δ depends on both a_1 and a_2 and not just on the ratio a_1/a_2 as in the KDV two-soliton solution (Hirota 1971).

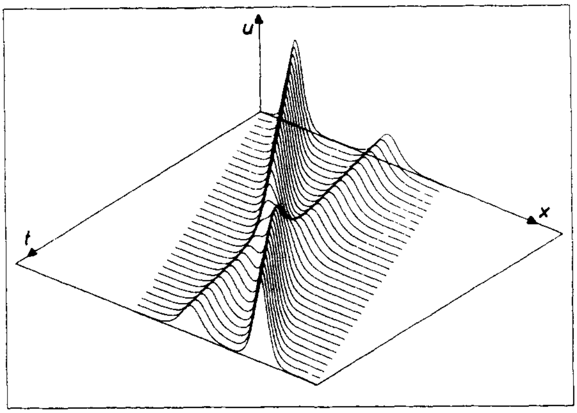

Figure 1. Numerical solution of the RLW equation (Eilbeck and McGuire 1976) with two-soliton initial conditions.

Since both the RLW equation and the modified RLW equation have the same single-soliton solution it is interesting to compare the numerical two-soliton solution of the RLW equation with the exact two-soliton solution of the modified RLW equation. Figure 1 shows the numerical two-soliton solution of the RLW equation and in figure 2 the numerical two-soliton phase shifts obtained from numerically integrating the RLW equation (with various choices of the amplitudes of the two colliding solitons) are compared with the analytic form (17) for the modified RLW equation. The fit is generally good, some slight discrepancy arising from the fact that the numerical RLW phase shifts for each soliton are not exactly equal and opposite (Eilbeck and McGuire 1976, Eilbeck *et al* 1976), where (17) with an appropriate choice of sign holds for both solitons in the modified RLW equation.

Finally we should point out that there is some lack of uniqueness in the derivation of (12). A choice of $D^3T+DT-D^2$ would have given the same dispersion relation (11). However this choice leads to a different equation for $u(x, t)$ and a different two-soliton phase shift which does not give good agreement with the numerically computed results of the RLW equation. Also two other equations similar to (13) have been suggested by

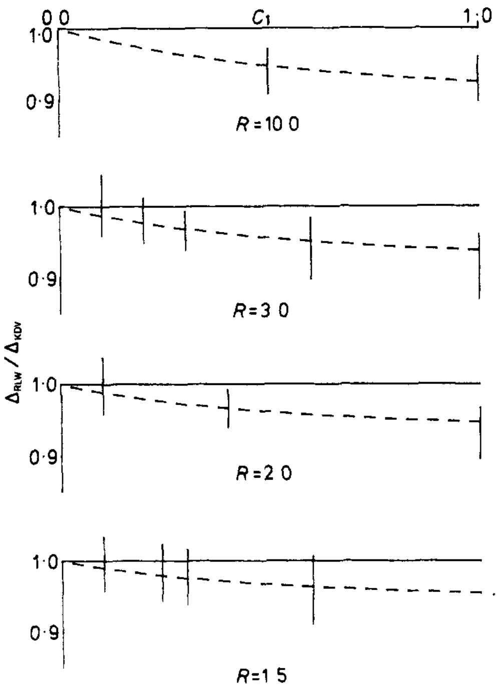

Figure 2. Plots of the RLW numerical two-soliton phase shifts and the modified RLW analytic two-soliton phase shifts as a ratio of the KDV two-soliton phase shifts. The numerical RLW phase shifts, with an estimated 2·5% error, are plotted as the vertical bars, and the analytic result as the broken curve. The amplitude of the largest soliton is $\frac{1}{4}C_1$ where $C_1 = a_1/(1-a_1^2)$ and R is the ratio of the two soliton amplitudes.

Ablowitz *et al* (1974) and Hirota and Satsuma (1976b). Both of these equations involve the transformation $u = (\ln f)_{xx}$ and hence their solitary-wave solution is different from that of the RLW equation.

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